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MET O 11 TECHNICAL NOTE NO 136

ON THE FILTERING OF SOUND WAVES FROM THE EQUATIONS FOR ATMOSPHERIC MOTION

– a sequel to Met O 11 Technical Note No 103

by

R N B SMITH

Met O 11
Meteorological Office
London Road
Bracknell
Berkshire
United Kingdom

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ON THE FILTERING OF SOUND WAVES FROM THE EQUATIONS OF ATMOSPHERIC MOTION

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1. INTRODUCTION

This note is a sequel to Met O 11 Technical Note No 103 by K M Carpenter. Therein several approximations which filter sound waves from the equations of motion were discussed. The hydrostatic approximation, while eliminating vertically propagating sound waves, seriously alters the dispersion relation, and hence the propagation properties of gravity waves even for quite low aspect ratios. This limits the general applicability of hydrostatic models to synoptic scales. Carpenter's note examined non-hydrostatic equations in which the anelastic approximation or an original, more accurate, generalised anelastic approximation filters compression waves.

Much of the discussion in Carpenter's note centred on the dispersion relation for the small amplitude waves which are described by the linearised equations. Carpenter's analysis contains an error in the non-oscillatory z variation of perturbation variables and the dependence of the dispersion relation on the associated scale height. We show in this note that the results of a correct calculation throw some light on the accuracy of anelastic approximations.

In § 2 we derive the dispersion relation for waves in the system described by the full equations. The anelastic approximation and Carpenter's generalisation of it are briefly introduced in § 3. The dispersion relations for gravity waves on an isothermal reference state is obtained for these two approximations and compared with that derived in § 2. In § 4 we discuss another "soundproofing" approximation introduced by Miller (1974) and named by him the quasi-non-hydrostatic approximation.

§ 2. SOUND AND GRAVITY WAVES

We write the full non-hydrostatic equations of motion in terms of the Exner function $P = (p/p_r)^{\frac{\gamma-1}{\gamma}}$ and the potential temperature θ :

$$\frac{D\underline{v}_H}{Dt} = -f\underline{k}\wedge\underline{v}_H - c_p\theta\nabla_H P + \underline{F} \tag{2.1}$$

$$\frac{Dw}{Dt} = -g - c_p\theta\frac{\partial P}{\partial z} \tag{2.2}$$

$$\frac{D\theta}{Dt} = \frac{Q}{c_p P} \tag{2.3}$$

$$\nabla.\underline{v} = \frac{1}{\theta}\frac{D\theta}{Dt} - \frac{1}{(\gamma-1)P}\frac{DP}{Dt} \tag{2.4}$$

Introducing an equilibrium state described by $\underline{v}_0 = \underline{0}$, $\theta_0(z)$ and $dp_0/dz = -g/c_p\theta_0$ and denoting small perturbations about this state by $\sim$, then the linearised equations for the perturbations are:

$$\frac{\partial \tilde{\underline{v}}_H}{\partial t} = -c_p \theta_0 \nabla_H \tilde{p} \tag{2.5}$$

$$\frac{\partial \tilde{w}}{\partial t} = g\frac{\tilde{\theta}}{\theta_0} - c_p\theta_0 \frac{\partial \tilde{p}}{\partial z} \tag{2.6}$$

$$\frac{\partial \tilde{\theta}}{\partial t} = -\tilde{w}\frac{d\theta_0}{dz} \tag{2.7}$$

$$\nabla . \tilde{\underline{v}}_H + \frac{\partial \tilde{w}}{\partial z} = \frac{g\tilde{w}}{c_0^2} - \frac{c_p\theta_0}{c_0^2}\frac{\partial \tilde{p}}{\partial t} \tag{2.8}$$

$c_0 = \sqrt{(\gamma-1)c_p\theta_0 p_0}$ is the speed of sound. We have ignored friction, diabatic effects and the Coriolis force in these linearised equations.

In order to obtain an analytic dispersion relation we have to assume an isothermal equilibrium state. It is then the case that c_0^2 = const. and $N_0^2 = (g/\theta_0)d\theta_0/dz = (\gamma-1)g^2/c_0^2$ is also constant. We seek a solution to equations (2.5)-(2.8) of the form $f_n(z)\, e^{i(k_1 x + k_2 y - \omega t)}$. Simple manipulations give us the following equation for $\tilde{w}$:

$$\frac{\partial^2 \tilde{w}}{\partial z^2} - \left(\frac{N_0^2}{g} + \frac{g}{c_0^2}\right)\frac{\partial \tilde{w}}{\partial z} + \left(\frac{\omega^2}{c_0^2} - K^2 + \frac{K^2 N_0^2}{\omega^2}\right)\tilde{w} = 0 \tag{2.9}$$

Assuming a z dependence for $\tilde{w}$ of the form $e^{i(m+i\mu)z}$ with m and μ real constants, (2.9) implies that

$$\mu = -\frac{1}{2}\left(\frac{N_0^2}{g} + \frac{g}{c_0^2}\right) = -\frac{\gamma g}{2c_0^2}$$

and $$\boxed{\omega^4} - (\boxed{K^2} + m^2 + \mu^2)c_0^2\omega^2 + N_0^2K^2c_0^2 = 0 \tag{2.10}$$

unless $\tilde{w} \equiv 0$. Define $\omega_s^2 = (K^2+m^2+\mu^2)c_0^2$ and $\omega_g^2 = N_0^2K^2/(K^2+m^2+\mu^2)$. If $\omega_g^2/\omega_s^2 \ll 1$, as is the case in the atmosphere for meteorologically important motions, then the dispersion relation (2.10) has the approximate solutions

$$\omega^2 \simeq \omega_g^2\left(1 + \frac{\omega_g^2}{\omega_s^2}\right) \quad \text{and} \quad \omega^2 \simeq \omega_s^2\left[1 - \frac{\omega_g^2}{\omega_s^2}\left(1 + \frac{\omega_g^2}{\omega_s^2}\right)\right] \tag{2.11}$$

The first solution corresponds to gravity waves and the second to sound waves. If the hydrostatic approximation is made in (2.2) the two terms marked $\boxed{\ }$ in (2.10) do not appear and the solution becomes

$$\omega^2 = N_0^2K^2/(m^2+\mu^2)$$

Thus sound waves are lost from the hydrostatic system but gravity waves will not be represented properly unless $k^2 \ll m^2, \mu^2$.

If $\tilde{w} \equiv 0$ we cannot deduce that (2.10) holds and we must proceed differently. Putting $\tilde{w} = 0$ in (2.5)–(2.8) we find $\tilde{\theta} = 0$, $\partial\tilde{p}/\partial z = 0$ and $(\omega^2 - k^2 c_0^2)\tilde{p} = 0$. The mode given by $\omega^2 = k^2 c_0^2$ is the Lamb wave which travels horizontally with the speed of sound and is not eliminated by making the hydrostatic approximation.

We note that the reciprocal scale height μ of the non-oscillatory z variation is given by $\mu = -\gamma g/2c_0^2$ whereas Carpenter (1978) claims $\mu = -g/c_0^2$. The negative value of μ implies an exponential increase with height for $\tilde{w}$ but it can be shown that any density weighted quantity, in particular the wave energy, remains bounded.

§ 3. ANELASTIC APPROXIMATIONS

We will not discuss anelastic approximations in detail. Readers should refer to Carpenter (1978) for a fuller treatment. Central to these approximations is the definition of a reference potential temperature $\bar{\theta}(z)$ and Exner variable $\bar{p}(z)$ with $d\bar{p}/dz = -g/c_p\bar{\theta}$.

Writing $\theta' = \theta - \bar{\theta}$ and $p' = p - \bar{p}$ we can approximate the full equations (2.1)–(2.4) by the set:

$$\frac{D\underline{v}_H}{Dt} = -f\underline{k}\wedge\underline{v}_H - c_p\bar{\theta}\nabla_H p' + \underline{F} \tag{3.1}$$

$$\frac{Dw}{Dt} = g\frac{\theta'}{\bar{\theta}} - c_p\bar{\theta}\frac{\partial p'}{\partial z} \tag{3.2}$$

$$\frac{D\theta'}{Dt} + w\frac{d\bar{\theta}}{dz} = \frac{Q}{c_p\bar{p}} \tag{3.3}$$

$$\nabla.\underline{v} = \frac{w}{\bar{\theta}}\frac{d\bar{\theta}}{dz} + \frac{gw}{(\gamma-1)c_p\bar{\theta}\bar{p}} \tag{3.4}$$

If $\bar{\theta}$ = const. we have the original anelastic approximation of Ogura and Phillips (1962).

Defining $\bar{\rho} = \frac{p_r}{R}\bar{p}^{\frac{1}{\gamma-1}}\bar{\theta}^{-1}$ it can be shown that (3.1)–(3.4) satisfy the conservation laws:

$$\frac{\partial\bar{\rho}}{\partial t} + \nabla.(\bar{\rho}\underline{v}) = 0$$

$$\frac{\partial(\bar{\rho}\underline{v}_H)}{\partial t} + \nabla.(\bar{\rho}\underline{v}_H\underline{v}) = -f\bar{\rho}\underline{k}\wedge\underline{v}_H - \nabla_H\hat{p} + \bar{\rho}\underline{F}$$

$$\frac{\partial(\bar{\rho}w)}{\partial t} + \nabla.(\bar{\rho}w\underline{v}) = -\bar{\rho}g\left(1 + \frac{1}{(\gamma-1)}\frac{p'}{\bar{p}} - \frac{\theta'}{\bar{\theta}}\right) - \frac{\partial\hat{p}}{\partial z}$$

$$\frac{\partial(\bar{\rho}\hat{E})}{\partial t} + \nabla.(\bar{\rho}\hat{E}\underline{v}) = \bar{\rho}(Q + \underline{v}.\underline{F}) + \frac{\partial \hat{p}}{\partial t}$$

where the modified pressure, $\hat{p} = p_r\left(\bar{p}^{\frac{\gamma}{\gamma-1}} + \frac{\gamma}{(\gamma-1)}\bar{p}^{\frac{1}{\gamma-1}}p'\right)$
and the modified enthalpy, $\hat{E} = \frac{1}{2}v_H^2 + \frac{1}{2}w^2 + c_p\hat{T} + gz$
and the modified temperature, $\hat{T} = \bar{p}\bar{\theta} + \bar{p}\theta' + p'\bar{\theta}$.

In order to derive an analytic dispersion relation for small amplitude waves we have to assume that the reference state is isothermal and that the equilibrium state which is perturbed is also isothermal. With these assumptions the dispersion relation is

$$\omega^2 = \frac{N_o^2 K^2}{\left[K^2 + m^2 + \left(\frac{g}{2c_o^2}\right)^2\right]} \tag{3.5}$$

and the non-oscillatory z variation of $\tilde{w}$ is $\propto e^{-\lambda z}$ where

$$\lambda = -\frac{1}{2}\left(\frac{g}{c_o^2} + \frac{2N_o^2}{g}\right) = \mu - \frac{N_o^2}{2g} \tag{3.6}$$

The error in this reciprocal scale height and part of the error in the dispersion relation for gravity waves (3.5) originate from the term $(w/\bar{\theta})d\bar{\theta}/dz$ in the approximated continuity equation (3.4). This term does not appear in Ogura and Phillip's equations where the reference state is isentropic rather than isothermal. We would be unwise, therefore, to base our discussion about the validity of Ogura and Phillip's anelastic approximation on the equation (3.5) and (3.6) above. However, by means of a scale analysis it can be shown (see Carpenter (1978)) that this approximation has limited accuracy.

A noteworthy characteristic of the anelastic approximation is the effect of filtering all compression waves, including Lamb waves.

Carpenter (1978) suggested a generalised anelastic approximation which, while eliminating all compression waves, does not leave out so many terms from the full equations. With $\bar{\theta}$, θ' etc having the same meaning as before, Carpenter's anelastic approximation is:

$$\frac{D\underline{v}_H}{Dt} = -f\underline{k}\wedge\underline{v}_H - c_p\theta\nabla_H p' + \underline{F} \tag{3.7}$$

$$\frac{Dw}{Dt} = g\frac{\theta'}{\bar{\theta}} - c_p\theta\frac{\partial p'}{\partial z} \tag{3.8}$$

$$\frac{D\theta'}{Dt} + w\frac{d\theta_o}{dz} = \frac{Q}{c_p P} \tag{3.9}$$

$$\nabla.\underline{v} = \frac{1}{\theta}\frac{D\theta}{Dt} + \frac{gw}{(\gamma-1)c_p\bar{\theta}\bar{p}} \tag{3.10}$$

Defining a modified density by $\hat{\rho} = \frac{p_r}{R}\bar{P}^{\frac{1}{\gamma-1}}\theta^{-1}$, the system described by equations (3.7)–(3.10) satisfies the conservation laws:

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla.(\hat{\rho}\underline{v}) = 0$$

$$\frac{\partial(\hat{\rho}\underline{v}_H)}{\partial t} + \nabla.(\hat{\rho}\underline{v}_H\underline{v}) = -f\hat{\rho}\underline{k}\wedge\underline{v}_H - \nabla_H\hat{p} + \hat{\rho}\underline{F}$$

$$\frac{\partial(\hat{\rho}w)}{\partial t} + \nabla.(\hat{\rho}w\underline{v}) = -g\hat{\rho}\left(1 + \frac{1}{(\gamma-1)}\frac{p'}{\bar{p}}\frac{\theta}{\bar{\theta}}\right) - \frac{\partial \hat{p}}{\partial z}$$

$$\frac{\partial(\hat{\rho}E)}{\partial t} + \nabla.(\hat{\rho}E\underline{v}) = \hat{\rho}(Q + \underline{v}.\underline{F}) + \frac{\partial \hat{p}}{\partial t}$$

where the modified pressure, $\hat{p} = p_r\left(\bar{P}^{\frac{\gamma}{\gamma-1}} + \frac{\gamma}{(\gamma-1)}\bar{P}^{\frac{1}{\gamma-1}}p'\right)$ and the enthalpy/unit mass, $E = \frac{1}{2}\underline{v}^2 + c_pT + gz$ with $T = \theta P$.

Again an analytic dispersion relation for small amplitude waves on an isothermal equilibrium/reference state can be found. It turns out to be:

$$\omega^2 = \frac{N_c^2K^2}{\left(K^2 + m^2 + \mu^2 - \frac{N_c^2}{c_c^2}\right)} \qquad (3.11)$$

The space-time variation of $\hat{w} \propto e^{i(k_1x + k_2y + mz - \omega t)}e^{-\mu z}$ with $\mu = -\gamma g/2c_c^2$ (as with the full equations). Sound and Lamb waves are not present.

Carpenter's generalised anelastic approximation is clearly more accurate than the ordinary anelastic approximation. Nevertheless, the propagation properties of gravity waves are somewhat distorted. The gravity wave solution to the full equations correct to $O(\omega_g^4/\omega_s^4)$ is given by (2.11). It can be written, to the same degree of accuracy, as

$$\omega^2 = \frac{N_c^2K^2}{\left\{K^2 + m^2 + \mu^2 - \frac{N_c^2}{c_c^2}\left(\frac{K^2}{K^2 + m^2 + \mu^2}\right)\right\}} \qquad (3.12)$$

Comparison of (3.11) with (3.12) immediately shows that the generalised anelastic approximation entails very little error if $K^2 \gg m^2 + \mu^2$, ie if the horizontal wavelength is much smaller than the vertical wavelength. This condition will not be satisfied by the modes resolved in a mesoscale forecast model. However, the condition $m^2 \gg \mu^2$ ensures that there is only a small error in the treatment of gravity waves whatever the relative magnitude of K^2. In terms of the vertical wavelength, h, this condition becomes $h \ll 100$ km. Such a restriction on the vertical scales of motion for which the generalised anelastic approximation is good is what Carpenter (1978) deduced by performing a scale analysis of the full equations. The deductions made from a scale analysis are of course far more

general than those made from a consideration of small amplitude plane waves on an isothermal basic state.

In the generalised anelastic approximation we replace $\frac{1}{(\gamma-1)P}\frac{DP}{Dt}$ in the continuity equation (2.4) by $\frac{1}{(\gamma-1)\bar{P}}\frac{D\bar{P}}{Dt} = \frac{-wg}{(\gamma-1)c_p\bar{\theta}\bar{P}}$ thus obtaining (3.10). A term proportional to Dp'/Dt is therefore omitted. Part of the term Dp'/Dt is $w\partial p'/\partial z$ which is approximately $wg\theta'/c_p\bar{\theta}^2$. Both Carpenter's scale analysis and the details of the calculation of the dispersion relation indicate that it is this omitted term proportional to θ' which is the main source of inaccuracy.

An attempt has been made to eliminate this source of error by replacing $\frac{1}{(\gamma-1)P}\frac{DP}{Dt}$ by $\frac{1}{(\gamma-1)P_h}\frac{DP_h}{Dt}$ where $\frac{\partial P_h}{\partial z} = -\frac{g}{c_p\theta}$ rather than by $\frac{1}{(\gamma-1)\bar{P}}\frac{D\bar{P}}{Dt}$ as in Carpenter's anelastic approximation. This approximation, while filtering sound waves, leaves linear gravity waves undistorted and retains Lamb waves. It also removes the restrictions deduced from Carpenter's scale analysis. However, conservation properties analogous to those holding for the anelastic approximations discussed above do not hold. Because of this serious fault it has not been pursued.

4. THE QUASI-NON-HYDROSTATIC APPROXIMATION

Miller (1974) and Miller and Pearce (1974) presented and discussed a set of sound-proofed equations for use in their modelling of cumulonimbus clouds. The approximations involved in this set of equations are of an essentially different nature to the anelastic approximation.

Miller's approximation, which he calls the quasi-non-hydrostatic approximation, is best approached through the use of pressure co-ordinates. The full equations (2.1)-(2.4), written with pressure as the vertical co-ordinate are:

$$\frac{D\underline{v}_H}{Dt} = -f\underline{k}\wedge\underline{v}_H - \chi\nabla_p\phi + \underline{F} \tag{4.1}$$

$$\frac{Dw}{Dt} = g(\chi-1) \tag{4.2}$$

$$\frac{DT}{Dt} = \frac{\omega}{c_p\rho} + \frac{Q}{c_p} \tag{4.3}$$

$$\nabla_p\cdot\underline{v}_H + \frac{\partial\omega}{\partial p} = \frac{D}{Dt}\ln\chi \tag{4.4}$$

where $\chi = \frac{1}{\rho}\left(-\frac{\partial\phi}{\partial p}\right)^{-1}$ assumed > 0, $\omega = Dp/Dt$,

$\phi = gh$, h being the height of the isobaric surface with pressure p,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\nabla_p + \omega\frac{\partial}{\partial p},$$

$$w = \frac{1}{g}\frac{D\phi}{Dt} = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \underline{v}\cdot\nabla_p h - \frac{\omega}{\rho g\chi} \tag{4.5}$$

Note that putting $\chi = 1$ gives us the hydrostatic equations.

To obtain the quasi-non-hydrostatic equations the following three approximations have to be made:

(i) write (4.2) as $$\frac{1}{\chi} = \left(1 + \frac{1}{g}\frac{Dw}{Dt}\right)^{-1} \simeq 1 - \frac{1}{g}\frac{Dw}{Dt}$$

assuming the vertical acceleration of a fluid particle is much less than g. We now have, in place of (4.2),

$$\frac{Dw}{Dt} = g\left(1 + \rho\frac{\partial\phi}{\partial p}\right) \tag{4.6}$$

(ii) put $\chi = 1$ in the horizontal momentum equation (4.1) and in the continuity equation (4.4).

(iii) replace Dw/Dt by $D\hat{w}/Dt$ where $\hat{w} = -\omega/\rho g$. Thus we approximate the expression (4.5) for w by ignoring the first two terms and putting $\chi = 1$ in the third.

The quasi-non-hydrostatic equations are therefore:

$$\frac{D\underline{v}_H}{Dt} = -f\,\underline{k}\wedge\underline{v}_H - \nabla_p\phi + \underline{F} \tag{4.7}$$

$$\frac{D}{Dt}\left(\frac{\omega}{\rho g}\right) = -g\left(1 + \rho\frac{\partial\phi}{\partial p}\right) \tag{4.8}$$

$$\frac{DT}{Dt} = \frac{\omega}{c_p\rho} + \frac{Q}{c_p} \tag{4.9}$$

$$\nabla_p.\underline{v} + \frac{\partial\omega}{\partial p} = 0 \tag{4.10}$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}.\nabla_p + \omega\frac{\partial}{\partial p}$$

and $\rho = p/RT$

The equations (4.7)-(4.10) look identical to the usual hydrostatic equations in pressure co-ordinates except for the non-zero approximation for the vertical acceleration on the lhs of (4.8). The geopotential, ϕ, can easily be shown to satisfy the diagnostic equation:

$$\nabla_p^2\phi + \frac{\partial}{\partial p}\left(g^2\rho^2\frac{\partial\phi}{\partial p}\right) = \nabla_p.\underline{G}_H + \frac{\partial G_3}{\partial p} \tag{4.11}$$

where $$\underline{G}_H = -\underline{v}.\nabla_p\underline{v}_H - \omega\frac{\partial\underline{v}_H}{\partial p} - f\,\underline{k}\wedge\underline{v}_H + \underline{F}$$

$$G_3 = -\underline{v}.\nabla_p\omega - \omega\frac{\partial\omega}{\partial p} - \rho g^2 + \frac{\omega}{\gamma p}\left[\omega + (\gamma-1)\rho Q\right]$$

The consistency of the three parts of this approximation is best proved by showing that (4.7)–(4.10) imply appropriate conservation laws. To this end we first write equations (4.7)–(4.10) in z co-ordinates using the variables of §2:

$$\frac{D\underline{v}_H}{Dt} = -f\,\underline{k}\wedge\underline{v}_H - \frac{c_p\theta}{\chi}\nabla_H P + \underline{F} \tag{4.12}$$

$$\frac{D\hat{w}}{Dt} = \frac{1}{\chi}\left(-g - c_p\theta\frac{\partial P}{\partial z}\right) \tag{4.13}$$

$$\frac{D\theta}{Dt} = \frac{Q}{c_p P} \tag{4.14}$$

$$\nabla.\underline{v} = \frac{1}{\theta}\frac{D\theta}{Dt} - \frac{1}{(\gamma-1)P}\frac{DP}{Dt} - \frac{D}{Dt}\ln\chi \tag{4.15}$$

where $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \underline{v}_H\cdot\nabla + w\dfrac{\partial}{\partial z}$

$\chi = -\dfrac{c_p\theta}{g}\dfrac{\partial P}{\partial z}$ (NB. this is the same χ as defined above)

and $\hat{w} = -\dfrac{c_p\theta}{g}\dfrac{DP}{Dt} = -\dfrac{1}{\rho g}\dfrac{D_p}{Dt}$

Note that w is only replaced by $\hat{w}$ when operated on by D/Dt ; the vertical advection still involves w. Defining a modified density by

$$\hat{\rho} = \frac{p_r}{R}P^{\frac{1}{\gamma-1}}\theta^{-1}\chi = \frac{-1}{g}\frac{\partial p}{\partial z}$$

we can deduce from (4.12)–(4.15) the following equations in flux form:

$$\frac{\partial\hat{\rho}}{\partial t} + \nabla.(\hat{\rho}\underline{v}) = 0$$

$$\frac{\partial(\hat{\rho}\underline{v}_H)}{\partial t} + \nabla.(\hat{\rho}\underline{v}_H\underline{v}) = -f\hat{\rho}\,\underline{k}\wedge\underline{v}_H - \nabla_H p + \hat{\rho}\underline{F}$$

$$\frac{\partial(\hat{\rho}\hat{w})}{\partial t} + \nabla.(\hat{\rho}\hat{w}\underline{v}) = -\rho g - \frac{\partial p}{\partial z}$$ (Note the true density $\rho = p/RT$ multiplying g)

$$\frac{\partial(\hat{\rho}c_p\theta P)}{\partial t} + \nabla.(\hat{\rho}c_p\theta P\underline{v}) = -g\hat{\rho}\hat{w} + \hat{\rho}Q$$

$$\frac{\partial(\hat{\rho}gz)}{\partial t} + \nabla.(\hat{\rho}gz\underline{v}) = g\hat{\rho}w$$

$$\frac{\partial(\frac{1}{2}\hat{\rho}\underline{v}_H^2)}{\partial t} + \nabla.(\tfrac{1}{2}\hat{\rho}\underline{v}_H^2\underline{v}) = -\underline{v}_H.\nabla p + \hat{\rho}\underline{v}.\underline{F}$$

$$\frac{\partial(\frac{1}{2}\hat{\rho}\hat{w}^2)}{\partial t} + \nabla.(\tfrac{1}{2}\hat{\rho}\hat{w}^2\underline{v}) = -g\rho\hat{w} - \hat{w}\frac{\partial p}{\partial z}$$

Defining a modified enthalpy/unit mass $\hat{E}$ by

$$\hat{E} = \tfrac{1}{2}\underline{v}_H^2 + \tfrac{1}{2}\hat{w}^2 + c_p T + gz$$

we find, by adding the last four equations, that

$$\frac{\partial(\hat{\rho}\hat{E})}{\partial t} + \nabla.(\hat{\rho}\hat{E}\underline{v}) = \hat{\rho}(Q + \underline{v}.\underline{F}) + g(w-\hat{w})\rho - g\rho\hat{w} - \underline{v}_H.\nabla p - \hat{w}\frac{\partial p}{\partial z}$$

$$= \hat{\rho}(Q + \underline{v}.\underline{F}) + \frac{\partial p}{\partial t} + \left[-\frac{Dp}{Dt} - g\rho\hat{w} + g\hat{\rho}(w-\hat{w}) + (w-\hat{w})\frac{\partial p}{\partial z}\right]$$

$$= \hat{\rho}(Q + \underline{v}.\underline{F}) + \frac{\partial p}{\partial t} \qquad \text{since } g\hat{\rho} = -\frac{\partial p}{\partial z} \text{ and } g\rho\hat{w} = -\frac{Dp}{Dt}.$$

Thus the quasi-non-hydrostatic equations are energetically consistent. We can also see that approximations (ii) and (iii), despite appearances, are inextricably linked: making one without the other destroys the enthalpy conservation. Miller (1974), who did not discuss the conservation properties of his equations, deduced that (ii) and (iii) have to be taken together by linearising the full equations and finding that the dispersion relation has higher order than the true result (2.10) unless approximations (ii) and (iii) are <u>both</u> made.

Equations (4.12)–(4.15) linearised about an isothermal equilibrium state become (ignoring Coriolis force, diabatic effects etc):

$$\frac{\partial \tilde{\underline{v}}_H}{\partial t} = -c_p\theta_0 \nabla_H \tilde{P} \tag{4.16}$$

$$\frac{\partial \tilde{w}}{\partial t} = g\frac{\tilde{\theta}}{\theta_0} - c_p\theta_0\frac{\partial \tilde{P}}{\partial z} \boxed{+ \frac{c_p\theta_0}{g}\frac{\partial^2 \tilde{P}}{\partial t^2}} \tag{4.17}$$

$$\frac{\partial \tilde{\theta}}{\partial t} = -\tilde{w}\frac{d\theta_0}{dz} \tag{4.18}$$

$$\nabla.\tilde{\underline{v}}_H = \frac{g\tilde{w}}{c_0^2} - \frac{\partial \tilde{w}}{\partial z} - \frac{c_p\theta_0}{c_0^2}\frac{\partial \tilde{P}}{\partial t} \boxed{- \frac{\partial}{\partial t}\left(\frac{\tilde{\theta}}{\theta_0} - \frac{c_p\theta_0}{g}\frac{\partial \tilde{P}}{\partial z}\right)} \tag{4.19}$$

Note the extra terms denoted $\boxed{}$ which appear in (4.16)–(4.19) but not in the linearised full equations (2.5)–(2.8). The dispersion relation is found to be

$$\omega^2 = \frac{N_0^2 K^2}{(K^2 + m^2 + \mu^2)} \tag{4.20}$$

where $\mu = -\gamma g/2c_0^2$ and $\tilde{w} \propto e^{i(k_1 x + k_2 y + mz - \omega t)}e^{-\mu z}$. We see that sound waves are filtered and that gravity waves are well represented. If $\tilde{w} \equiv 0$ equation (4.20) does not hold and we must proceed differently to obtain the Lamb wave solution. Putting $\tilde{w}=0$ in (4.16)–(4.19) we deduce that $\tilde{\theta}=0$, $\tilde{P} \propto e^{i(k_1 x + k_2 y - \omega t)}e^{-\mu z}$, $\mu = \omega^2/g$ and

$$\omega^4 + \frac{g^2}{c_0^2}\omega^2 - k^2 g^2 = 0 \qquad (4.21)$$

The product of the two roots of (4.21) is $-k^2g^2 < 0$ so one root is positive and the other negative. The positive root is given by

$$\omega^2 = \frac{-g^2}{2c_0^2} + \frac{1}{2}\sqrt{\frac{g^4}{c_0^4} + 4g^2k^2} \qquad (4.22)$$

If $\frac{4k^2c_0^4}{g^2} \ll 1$, ie $|k| \ll \frac{g}{2c_0^2}$ or $L = 2\pi/|k| \gg$ 140 km, we can expand the square root in (4.22) to obtain the approximate solution $\omega^2 \simeq k^2c_0^2$. So for horizontal scales much bigger than 140 km the Lamb wave is not appreciably distorted by the quasi-non-hydrostatic approximation.

It is hard to believe that Lamb waves are meteorologically important except possibly those with very long wavelengths and low frequencies. The Lamb waves on these scales are virtually undistorted by the quasi-non-hydrostatic approximation. It is in any case usual to filter Lamb waves from numerical models using pressure co-ordinates by imposing a boundary condition $\omega = 0$ on a surface $p = p_1$. Applying such a boundary condition also eliminates the instability represented by the negative root of (4.21).

In so far as we can deduce the range of validity of an approximation from the dispersion relation for small amplitude waves in the free atmosphere we can say that the quasi-non-hydrostatic approximation is good on all scales of meteorological interest. The scale analysis presented by Miller (1974) confirms this conclusion for the unlinearised equations.

For this analysis a height field for the undisturbed atmosphere is defined by $dh_0/dp = -1/\rho_0(p)\,g$. The total height field is then written as $h = h_0 + h_1$. Let U, Ω, H' be typical values of $|v_H|, \omega, h_1$ and let L, H, P be characteristic horizontal, vertical and pressure scales. We have

$$\chi = 1 + O\left(\frac{p_1}{p_0}, \frac{H'}{H}\right).$$

The continuity equation gives $U/L \sim \Omega/P$ and so $D/Dt \sim U/L$. The horizontal equation of motion then implies that $U^2 \sim gH'$. Equation (4.2) can be written as

$$\underbrace{\frac{1}{g}\frac{D}{Dt}\left(\frac{\partial h'}{\partial t} + v \cdot \nabla_p h'\right)}_{\sim (H'/L)^2} - \underbrace{\frac{1}{g}\frac{D}{Dt}\left(\frac{\omega}{\rho g \chi}\right)}_{\sim (H'/L)(H/L)} = \underbrace{\chi - 1}_{\sim \left(\frac{p_1}{p_0}, \frac{H'}{H}\right)}$$

If $H \lesssim L$ and only terms of lowest order in H'/H are retained in (4.1)-(4.4) we arrive at the quasi-non-hydrostatic set (4.7)-(4.10). Thus the approximation is accurate to the extent that $H'/H \ll 1$, ie $H' \ll 10$ km. This condition is likely to be well satisfied for mesoscale systems.

A numerical model utilizing the quasi-non-hydrostatic approximation to filter sound waves would have to use pressure co-ordinates - the equations are very

complicated when written in z or σ co-ordinates. When using pressure co-ordinates the lower boundary condition is complicated by the fact that the earth's surface is a moving surface in the pressure co-ordinate system. This makes the realistic treatment of orography particularly difficult. Therefore the quasi-non-hydrostatic approximation may not be appropriate for a mesoscale forecast model or for simulations of flow over orography. It is well suited for its original use in cumulonimbus modelling since the use of pressure co-ordinates simplifies some of the thermodynamical calculations when moisture is included (Miller (1974)).

Acknowledgement

I am grateful to Dr K M Carpenter for several useful discussions during the preparation of this note and for his comments on a first draft.

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